Dynamic Quality and Advertising Decisions Considering Product Recalls in Supply Chain

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Abstract Under the trend of economic globalization and lean production, how to cope with the costs in the supply chain is a core issue to be considered in the operation and management of enterprises. In this paper, we construct a supply chain system consisting of a supplier and a manufacturer who buys components from the supplier to produce finished products and then sells them to consumers. Considering the potential product recalls and the existence of cost-sharing contracts among supply chain members, we constructed game models for the non-cooperative and cooperative scenarios respectively, and explored how supply chain members adjusted their advertising media and quality investments to cope with the product recalls in both scenarios; we also explored the coordinating effects of the cost-sharing contracts on the supply chain through the comparisons of the two scenarios. The study finds that: Manufacturers can effectively incentivize suppliers to invest more in primary quality and improve product quality through cost-sharing contracts. The effectiveness of cost-sharing contracts in coordinating the supply chain is affected by the likelihood of recalls and the extent of recall damage, and cooperation can only be achieved when both are relatively small.

Keywords Product recall; Product quality; Cost sharing; Differential games

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Introduction

In recent years, recalls of defective products occur frequently. Toyota's pedal failure incident, Samsung's phone battery explosion incident, IKEA chest of drawers injury children incident and so on, every incident has caused widespread concern in society. Product quality management and recall regulatory issues have been troubling the public. In 2023, Chinese government issued the "Outline of the construction of a strong quality country" clearly pointed out that we must " improve the product recall management system and mechanism, strengthen the technical support of recall, strengthen the defective product recall management, and effectively safeguard the rights and interests of consumers and social and public safety". The General Administration of Market Supervision (GAMS) adheres to the principle of integration and development, strengthens the regulation of recalls, and introduces a number of policies to encourage enterprises to carry out the practice of safety design and promote the high-quality development of the industry. Product recalls are prevalent in electronic and electrical appliances, and automotive industries, and by the end of 2022 China had implemented a cumulative total of 2,628 automobile recalls involving 95,87,000 vehicles; and a cumulative total of 4,114 consumer product recalls involving 90,238,000 products, with electronic and electrical consumer products accounting for as much as 83.8% of the total.

With defective product recalls increasing year after year, the related issues have attracted the attention of scholars. Previous studies have mainly explored the following three aspects: (1) The consequences and internal mechanism of product recall. Product recall not only leads to the direct costs caused by collecting, replacing or repairing defective products in the short term, but also damages product goodwill, corporate image, and leads to a decline in sales in the long term. (Liu et al. 2017). From the perspective of consumers, product recall can significantly affect consumers' purchase intention (Wei et al. 2016). Product recall leads to an increase in investor expectations of enterprise costs, affecting investors' confidence, which then triggers divestment or selling stock, ultimately leading to the loss of stock returns (Shah et al. 2017). (2) Boundary conditions affecting product recalls. Government regulation, social opinion and etc. are the main external factors affecting corporate recall (Wang et al. 2017; Wei et al. 2016). From the perspective of internal corporate perspective, the severity of product quality crisis affects the speed of corporate recall, and it will strengthen the speed of corporate product recall if product defects affect consumers' personal safety(Eilert et al.2017). (3) Post-recall responses. Borah et al. (2016) shows that at the post-crisis stage, enhanced investments in advertising can mitigate the damage caused by lack of consumer trust. Lu et al. (2020) consider the different roles of manufacturers' and retailers' advertisements and use differential games to study manufacturers' brand advertisements strategies and retailers' promotional advertisement strategies, finding that manufacturers enhance their investments in advertisements at the onset of a crisis, while retailers always reduce their advertisement investments. Navas et al. (2021) investigate the synergistic effect between advertising and quality improvement and find that pre-crisis quality improvement not only contributes to the accumulation of pre-crisis goodwill, but also helps to recover goodwill after the crisis. Mukherjee et al. (2023) studied firms' equilibrium pricing, advertising, and quality investment decisions during the recall phase and subsequent recovery phase. While most of the above studies are based on the when and after stages of product recalls, our paper focus on the pre-recall stage. Because time is of the essence in product recalls, a timely response is of great importance in controlling the public opinion on product recalls as well as maintaining product goodwill and corporate image. In September 2020, Guangzhou Automobile Honda had a relatively small-scale recall event, with 10,386 units recalled, and the corporate announcement showed that the recall began to be implemented three days later, which was a quick action, and the handling of the recall event was positively recognized by the market.

Most manufacturing companies can face product recalls, even if not within the organization, problems can occur at some point in the supply chain leading to a recall event. Product recalls occur when a product threatens consumer safety or fails to meet specific standards that may be mandated or spontaneously adopted by the industry (Cleeren et al. 2017). In 2010, Toyota recalled 2.3 million vehicles, the stock price fell 22% in two weeks (Lister. 2010), and the reliability rating of the product dropped from 95% to 72% (Dong et al. 2021), the recall was due to a problem with the manufacturing of the supplier of the gas pedals; in 2016, Samsung recalled the Galaxy Note 7 series of cell phones, which caused Samsung's high-end cell phone market share from 35% to 17% (Pressman. 2017), with the recall attributed to problems with the battery supplier's manufacturing; and in 2018, Tesla announced a recall of 123,000 vehicles, with the stock price dropping 5.1% in just four days (Kim. 2018), with the recall attributed to problems with the supplier's manufacturing of the power steering components. The quality investment of each link in the supply chain jointly affects product quality, and any problem in any link results in a product recall, which will lead to a decline in consumer confidence in product quality, a decline in market demand, and thus affect the profitability of the enterprise. Therefore, this paper incorporates the quality investment of upstream suppliers into the product quality dynamics, the product quality level is affected by the quality investments of suppliers and manufacturers.

In view of the above, we consider a supply chain consisting of a supplier and a manufacturer, where the manufacturer buys components from the supplier to produce a product and then sells it to consumers. There are potential component defects that lead to product recalls and damage product goodwill, which in turn affects market demand and reduces firm revenue. In this paper, we develop a dynamic Stackelberg game model based on product quality and product goodwill, and investigate how manufacturer and supplier consciously inject resilience into their operations by formulating advertising and quality strategies to prepare for product recall, and how the two parties in the supply chain will coordinate the costs associated with product recall. In the Stackelberg game, the supplier is the leader and the manufacturer is the follower. The supplier first publishes its primary quality investments, and then the manufacturer decides its secondary quality investments based on the supplier's primary quality investments. Next, the manufacturer develops an advertising media investments decision with known quality status

The rest of the paper is organized as follows: Section 2 describes the studies in the related literature. Section 3 describes the model construction. Section 4 describes the model solution and correlation analysis. Section 5 performs the numerical analysis. In Section 6, we conclude the paper and provide an outlook on future research directions.

Literature review

Our research involves two main research directions: supply chain coordination management and product recall management.

Literature related to supply chain coordination management

Designing appropriate coordination contracts to improve supply chain performance has been of wide interest to scholars. Yoo and Cheong (2018) studied the effect of coordination contracts on supply chain product quality in static games; Buratto et al. (2019) investigated the impact of cooperative advertising contracts on supply chain product quality in dynamic games, and there is also a dynamic game literature on cooperative quality management through revenue-sharing contracts (El Ouardighi,2014)，Yong long (2019) used an electric vehicle battery replacement station as an example to explore how the channel of a battery replacement station with lease-perishable characteristics can be managed through appropriate wholesale price and repurchase price contracts for coordination. Mohammad et al. (2022) introduce revenue sharing contract in media supply chain based on data mining and multi-criteria decision making method, and find that revenue sharing contract leads to the improvement of actors' performance quality. Ma et al. (2020) construct a green supply chain system consisting of manufacturers and retailers, and study the cost sharing contract under uncertain information in the supply chain coordination problem and found that the profits of manufacturers and retailers in the cost-sharing case are higher than those in the distributed supply chain case. Liu et al. (2018) constructed a collaborative contingency inventory model to cope with supply disruptions and found that under the cost-sharing mechanism, the profits of manufacturers are smaller than those before coordination, while the profits of suppliers are larger than those before coordination, but the total profits of the whole supply chain become larger.

Literature related to product recall management

Product recall has a significant impact on supply chain performance, so supply chain decisions that consider product recall has received extensive attention: The literatures on supply chain performance from the perspective of product recall is divided into two main areas: First, the probability of product recall occurring or the cost of product recall can be reduced by establishing a supply chain traceability system or improving traceability capabilities. Piramuthu et al. (2013) Based on RFID technology to monitor the degree of defective food hazards and contamination source localization in perishable food supply chain network to reduce the cost of corporate recall. Epelbaum and Martinez (2014) found through empirical research that supply chain product traceability system can reduce the cost of product recall and improve the performance of the enterprise. Dai et al. (2015) through supply chain traceability system design, the information of each unit/batch of products or components can be tracked and traced to reasonably allocate the product recall cost, stimulate the supply chain enterprises to maintain consistent interests, and optimize the profit of the whole traceability chain. The second is to reduce the impact of product recalls on supply chain performance through the design of contracts between supply chain members. Dai et al. (2015) study the product recall problem in a secondary supply chain consisting of a manufacturer and two suppliers and find that a profit-sharing contract improves the economic profit of supply chain members. Chakraborty et al. (2023) study the impact of product recall on supply chain performance by constructing a manufacturer and supplier insurance contract model to study the changes in quality and pricing strategies before and after product recall, and found that it is always better for manufacturers to bear more cost related costs after product recall. The essence of insurance contracts is cost sharing, which improves the overall performance of the supply chain by sharing the costs associated with product recall. Insurance contracts have also received extensive attention from scholars in enhancing enterprise cost management. Wang and Luo (2015) study the optimal insurance contract design and ordering decisions of supply chain members under a capital constraint scenario. Dong and Tomlin (2012) study the relationship between business interruption insurance and operational metrics and find that insurance increases the marginal value of inventory and the overall value of emergency purchases. Serpa and Krishnan (2016) find that insurance contract can reduce the free-rider problems and increase the efficiency of cost management.

Model formulation

Consider a supply chain consisting of a manufacturer (M) and a supplier (S), where the manufacturer buys specific parts from the supplier to produce finished products and sells them to consumers. Manufacturer's setup for recalling products due to defective parts from suppliers is reasonable. For example, in September 2020, GAC Honda had a relatively small recall of 10,386 vehicles due to a defective voltage converter manufactured by a supplier, unrelated to the company's own manufacturing process. A product recall occurs at any random time τ in an infinite planning horizon $\left[0,\infty\right)$. The infinite planning horizon is divided by τ into two stages: Pre-recall stage $\left[0,\tau\right]$ and post- recall stage $\left[\tau,\infty\right)$. In this paper, index $j\in\left\{1,2\right\}$ are used to denote the pre-recall stage ($j=1$) and the post-recall stage ($j=2$), respectively. According to the settings of Lu and Navas (2021) and Mukherjee and Chauhan (2021), define a stochastic process $\left\{\Gamma(t):t>0\right\}$ to represent the stages before and after a product recall, where $\Gamma(\cdot)=1$ indicates the pre-recall stage and $\Gamma(\cdot)=2$ indicates the post-recall stage. $\chi(t)$ indicates the likelihood of a product recall occurring at a random time, i.e. $\lim\limits_{\Delta t\to 0}\dfrac{P[\Gamma(t+\Delta t)=2\mid\Gamma(t)=1]}{\Delta t}=\chi(t)$. Similar to the setting in Rubel (2018), this paper assumes that product recalls occur only once, and τ following an exponential distribution which expectation is $\dfrac{1}{\lambda}$, i.e. $\chi(t)=\lambda$.

Manufacturer collaborate supplier on quality management to prevent product recalls and improve customer satisfaction. For example, Toyota Motor in total quality management, involves suppliers in the product development process and supervises every process in product development to control the overall quality of the product (Mukherjee et al. 2023). Based on the upstream and downstream relationships of supply chain members, this paper names the supplier's quality investments as primary quality investments $u(t)$, Manufacturer's quality investments are named secondary quality investments $v(t)$. Product quality is a communal attribute of suppliers and manufacturers investments, The quality of the product at the moment t is $Q(t)$ and satisfies the following dynamic process:

$$\dot{Q}(t) = \alpha u(t) + \beta v(t) - \delta Q(t), \quad Q(0) = Q_0. \tag{1}$$

α, β are positive coefficients, indicating the effectiveness of primary investments and secondary quality investments on product quality improvement, respectively. δ indicates the depreciation rate of product quality. $Q_0 = 0$ is the initial quality level of the product.

In the information age, the advertising media, as a bridge of information dissemination to the public, conveys objective information on the causes, effects, and countermeasures of the recall event to the public at each stage, and plays a role in rallying people's faiths. Helena et al. (2020) found a 43% decrease in advertising effectiveness during product recall compared to pre-crisis. Mukherjee et al. (2023) suggest that product recall reduce the effectiveness of advertising investments on goodwill improvement. Chauhan et al. (2021) argue that product recall accelerate the natural rate of goodwill decay. In this paper, we denote the manufacturer's advertising investments at moment t by $A(t)$. In order to characterize the change of product goodwill before and after recall, with reference to Lu et al. (2020), the dynamic process of product goodwill $G(t)$ is defined as:

$$\dot{G}(t) = \begin{cases} k_1 A(t) + bQ(t) - \varepsilon_1 G(t), & 0 < t < \tau, \ G(0) = G_0, \\ k_2 A(t) + bQ(t) - \varepsilon_2 G(t), & t > \tau, \ G(\tau^+) = (1-\phi)G(\tau^-), \end{cases} \tag{2}$$

where b represents the impact of product quality on goodwill. ε_1 and ε_2 are the decay rates of goodwill at the pre-recall and post-recall stages, respectively. k_1 and k_2 denote the effects of the manufacturer's advertising investments on goodwill in the two stages. The pre-recall stage is more effective than the post-recall stage in terms of advertising investments, and the natural decay rate of goodwill is slower, i.e. $k_1 > k_2$, $\varepsilon_1 < \varepsilon_2$. This means that manufacturers wanting to achieve the same level of goodwill at the post-recall stage as they did at the pre-recall stage will have to invest more in advertising. $G_0 = 0$ is the initial value of the perceived quality of the product, and the effect of a product recall on the state of goodwill is represented by a jump in the state variable, namely $G(\tau^+) = (1-\phi)G(\tau^-)$, where $G(\tau^-)$ and $G(\tau^+)$ are the product goodwill before and after the recall moment τ, ϕ denotes the shock to goodwill from a product recall, with a larger shock implying a larger decline in goodwill. The larger the product recall shock, the larger the corresponding $k_1 - k_2$, $\varepsilon_2 - \varepsilon_1$.

With reference to De Giovanni (2019), this paper defines the market demand for a product as a linear function of price and goodwill. Demand relies negatively on the exogenous price p and positively on the goodwill status of the product, so the dynamic process of demand is denoted as:

$$D(t) = a - p + \sigma G(t), \tag{3}$$

where a denotes market potential, and σ denote the sensitivity of demand to goodwill, respectively.

By the eight dimensions of product quality that Garvin (1988) pioneered in measuring product quality: Consistency is the ability of a product or service to meet a specific standard. Consistency can be viewed as a protective measure taken by the supplier in response to a product recall, while quality improvement is a revised measure taken by the supplier due to product defects. Referring to Chakraborty et al. (2023), this paper assumes that the supplier's primary quality investments cost at time t is the sum of the consistency quality cost and the quality improvement cost, denoted by $C_S(t)$:

$$C_S(t) = \frac{1}{2}\mu u^2(t) + \frac{1}{2}\theta(1-\lambda)u^2(t). \tag{4}$$

$\frac{1}{2}\theta(1-\lambda)u^2(t)$ denotes the consistency quality cost and θ is the consistency quality cost coefficient. At the post-recall stage $\lambda=0$, the consistency quality cost is $\frac{1}{2}\theta u^2(t)$; At the pre-recall stage $\lambda\in(0,1]$, the greater the likelihood of a product recall occurring, the smaller the cost of consistency quality, implying that the supplier is less concerned about the product recall. $\frac{1}{2}\mu u^2(t)$ denotes the quality improvement cost and μ is the quality improvement cost coefficient. Similarly, the manufacturer's secondary quality investments costs and advertising investments costs are defined in quadratic form, i.e. $\frac{1}{2}\eta v^2(t)$ and $\frac{1}{2}hA^2(t)$. η, h are the cost parameters of the manufacturer's secondary quality investments and advertising investments, respectively.

We assumed that the manufacturer and the supplier are rational and that both seek to maximize the profits of their respective firms. The profit of the supplier at moment t in the two phase denote by $\pi_{S1}(t)$ and $\pi_{S2}(t)$, respectively; The profit of the moment t manufacturer at the two stages denoted by $\pi_{M1}(t)$ and $\pi_{M2}(t)$, respectively. The specific expressions are as follows:

$$\begin{cases}\pi_{S1}(t)=wD(\mathrm{t})-\frac{1}{2}\theta(1-\varphi)(1-\lambda)u^2(t)-(1-\varphi)\frac{1}{2}\mu u^2(t),0<t<\tau,\\ \pi_{S2}(t)=wD(\mathrm{t})-\frac{1}{2}(1-\varphi)\theta u^2(t)-(1-\varphi)\frac{1}{2}\mu u^2(t),t>\tau,\end{cases} \tag{6}$$

$$\begin{cases}\pi_{M1}(t)=(p-w)D(\mathrm{t})-\frac{1}{2}\eta v^2(t)-\frac{1}{2}hA^2(t)-\varphi\frac{1}{2}\mu u^2(t)-\varphi\frac{1}{2}\theta(1-\lambda)u^2(t),0<t<\tau,\\ \pi_{M2}(t)=(p-w)D(\mathrm{t})-\frac{1}{2}\eta v^2(t)-\frac{1}{2}hA^2(t)-\varphi\frac{1}{2}\mu u^2(t)\varphi\frac{1}{2}\theta u^2(t),t>\tau。\end{cases} \tag{7}$$

where φ is the cost-sharing ratio. Because the brunt of a product recall affects the manufacturer that sells the product, manufacturers have an incentive to work with suppliers in order to respond and address the incident as quickly as possible, covering a portion φ of the costs of supplier. When $\varphi=0$, there is no cost-sharing contract between the manufacturer and the supplier, and the supplier alone bears for all quality improvement costs and the product recall costs; when $\varphi\in(0,1)$, there is a cost-sharing contract between the supplier and the manufacturer. Table 1 summarizes all the parameters involved in this paper and their meanings.

Table 1: Parameters and their meanings

parameters	Hidden meaning
α	Effectiveness of primary quality efforts
β	Effectiveness of secondary quality efforts
δ	Natural decay rate of product quality
k_1	The Impact of Advertising Investment on Goodwill in the Pre-Recall Stage
ε_1	Natural decay of product goodwill in the pre-recall phase
k_2	The Impact of Advertising Investment on Goodwill in the Post-Recall Stage
ε_2	Natural decay of product goodwill in the post-recall phase market potential
b	The impact of product quality on goodwill. Market Potentials

a	Demand sensitivity to goodwill
σ	Product recall options
λ	Product Recall Shock
ϕ	Supplier quality improvement component cost factor
μ	Supplier consistency quality component cost factor
θ	Manufacturer's cost factor for secondary quality inputs
η	Manufacturer's advertising input cost factor
h	
φ	Proportion of costs that manufacturers help suppliers bear of a product recall

Model Analysis

This section uses the inverse-order induction method to solve for the optimal quality and advertising investments of supply chain members in cooperative and non-cooperative scenarios. The proof procedure of the propositions and Corollaries are shown in the Appendix.

Non-cooperation scenario (N)

In the non-cooperative scenario, manufacturers and suppliers make their decisions independently and there is no cost-sharing contract between the members, i.e. $\varphi = 0$ First solve for the manufacturer's optimal quality and advertising investments decision at the post-recall stage, and then solve for the supplier's and manufacturer's optimal quality and advertising investments decisions at the pre-recall stage.

Post-recall stage for products in non-cooperative scenario

The optimal control problems for the manufacturer and supplier at this stage are:

$$\begin{cases} \max\limits_{v(t),A(t)} \int_{\tau}^{\infty} e^{-rt}((p-w)D(t) - \frac{1}{2}\eta v^2(t) - \frac{1}{2}hA^2(t))dt \\ s.t. \dot{G}(t) = k_2 A(t) + bQ(t) - \varepsilon_2 G(t),\ G(\tau^+) = (1-\phi)G(\tau^-), \\ \dot{Q}(t) = \alpha u(t) + \beta v(t) - \delta Q(t),\ Q(0) = Q_0, \end{cases} \tag{8}$$

$$\begin{cases} \max\limits_{u(t)} \int_{\tau}^{\infty} e^{-rt}(wD(t) - \frac{1}{2}\mu u^2(t) - \frac{1}{2}\theta u^2(t))dt \\ s.t. \overset{g}{G}(t) = k_2 A(t) + bQ(t) - \varepsilon_2 G(t),\ G(\tau^+) = (1-\phi)G(\tau^-), \\ \overset{g}{Q}(t) = \alpha u(t) + \beta v(t) - \delta Q(t),\ Q(0) = Q_0. \end{cases} \tag{9}$$

Applying the dynamic planning approach, it can be seen that the Hamilton-Jacobi-Bellman (HJB) equations for the manufacturer and the supplier at the post-recall stage of the product recall are:

$$\begin{aligned} rV_{M2}^{N}(G,Q) = \max_{v,A}[(p-w)(a-p+\sigma G) - \frac{1}{2}\eta v^2 - \frac{1}{2}hA^2 + \frac{\partial V_{M2}^{N}}{\partial G}(k_2 A + bQ \\ -\varepsilon_2 G) + \frac{\partial V_{M2}^{N}}{\partial Q}(\alpha u + \beta v - \delta Q)], \end{aligned} \tag{10}$$

$$rV_{S2}^{N}(G,Q)=\max_{u}[p(a-p+\sigma G)-\frac{1}{2}\mu u^2-\frac{1}{2}\theta u^2+\frac{\partial V_{S2}^{N}}{\partial G}(k_2A+bQ-\varepsilon_2 G)$$
$$+\frac{\partial V_{S2}^{N}}{\partial Q}(\alpha u+\beta v-\delta Q)], \quad (11)$$

where superscript N denotes a non-cooperative scenario, V_{M2}^{N}, V_{S2}^{N} respectively denote the optimal value functions of the manufacturer and supplier at the post-recall stage, and satisfy the boundary condition $\lim_{t\to\infty} re^{-rt}V_{M2}^{N}=0$, $\lim_{t\to\infty} re^{-rt}V_{S2}^{N}=0$. The feedback solution of the Stackelberg game in the non-cooperative scenario is obtained by solving the HJB equations (10) and (11) , Proposition 1 shows the firm's equilibrium decision and optimal value function at the post-recall stage in the non-cooperative scenario.

Proposition 1 In the non-cooperative situation, the optimal advertising and secondary quality investments of the manufacturer and the optimal primary quality investments of the supplier at the post-recall stage are:

$$A_2^{N}=\frac{\sigma(p-w)\ k_2}{(r+\varepsilon_2)h}, \quad (12)$$

$$v_2^{N}=\frac{b\beta\sigma(p-w)}{(r+\varepsilon_2)(r+\delta)\eta}, \quad (13)$$

$$u_2^{N}=\frac{w\sigma b\alpha}{(r+\varepsilon_2)(r+\delta)(\mu+\theta)}. \quad (14)$$

The optimal value function of manufacturer and supplier at the post-recall stage are:

$$V_{M2}^{N}=\frac{\sigma(p-w)}{r+\varepsilon_2}G+\frac{\sigma b(p-w)}{(r+\varepsilon_2)(r+\delta)}Q+F_1, \quad (15)$$

$$V_{S2}^{N}=\frac{w\sigma}{r+\varepsilon_2}G+\frac{w\sigma b}{(r+\varepsilon_2)(r+\delta)}Q+F_2, \quad (16)$$

where $Q^{N}=\left(Q_0-Q_\infty^{N}\right)e^{-\delta t}+Q_\infty^{N}$, $G^{N}=(G_0-G_\infty^{N})e^{-\varepsilon_2 t}+G_\infty^{N}$ are the optimal paths for product quality and goodwill, respectively. The steady states of product optimal quality and goodwill are $Q_\infty^{N}=\frac{b\sigma(\alpha^2w\eta+\beta^2\mu p-\beta^2\mu w+\beta^2p\theta-\beta^2\theta w)}{\delta(r+\varepsilon_2)(r+\delta)(\mu+\theta)\eta}$, $G_\infty^{N}=\frac{Q_\infty^{N}bhr+Q_\infty^{N}bh\varepsilon_2+\sigma k_2^2p-\sigma k_2^2w}{\varepsilon_2(r+\varepsilon_2)h}$. The expressions of F_1, F_2 are shown in the appendix due to their complexity.

The equilibrium strategies in Proposition 1, such as manufacturers' advertising investments, secondary quality investments, and suppliers' primary quality investments, are independent of the likelihood of a product recall. In other words, the decision of these enterprises are not affected by the change in the likelihood of a recall. The reason for this is that at this stage, recalls have already occurred and decisions are based on existing facts and status quo. Further observation reveals that all three investment strategies increase incrementally. The positive utility of product goodwill for consumer demand increases as consumers become more concerned about product goodwill. In this context, suppliers and manufacturers pay more attention to product quality improvement and tend to invest more quality efforts to improve product quality. At the same time, manufacturers are willing to increase their advertising investment to improve product goodwill, even if this means higher marketing costs. This dual effect - improved product quality and increased marketing investment - together drive further improvements in product goodwill.

Pre-recall stage for products in non-cooperative scenario

Although the firm had expected the probability of a product recall to be λ, but the moment τ when the recall occurs is still uncertain. Referring to Mukherjee et al. (2023), suppliers and manufacturers seek to maximize their expected profit at this stage. The expected profit of the supplier and the manufacturer is the average of the profit at the pre-recall stage and the post-recall stage, using $\Pi_{S}^{N}(t)$ and $\Pi_{M}^{N}(t)$ to denote the expected profit of the supplier and the manufacturer, respectively:

$$\begin{cases}\Pi_{S1}^{N}(t)=E_t[\int_0^{\tau}e^{-rs}\pi_{S1}(s)ds+e^{-r\tau}J_{S2}^{N}(\tau)],\\ \Pi_{M1}^{N}(t)=E_t[\int_0^{\tau}e^{-rs}\pi_{M1}(s)ds+e^{-r\tau}J_{M2}^{N}(\tau)],\end{cases}\tag{17}$$

where $J_{M2}^{N}(\tau)=\int_{\tau}^{\infty}e^{-rt}\pi_{M2}(t)dt$ and $J_{S2}^{N}(\tau)=\int_{\tau}^{\infty}e^{-rt}\pi_{S2}(t)dt$ denote the present value of the manufacturer's and supplier's profits at the moment τ , respectively. Equation (12) can be transformed according to the integration by parts and algebraic operations:

$$\begin{cases}\Pi_{S1}^{N}(t;\lambda)=\int_0^{\infty}e^{-(r+\lambda)t}(\pi_{S1}(t)+\lambda J_{S2}^{N}(t))dt,\\ \Pi_{M1}^{N}(t;\lambda)=\int_0^{\infty}e^{-(r+\lambda)t}(\pi_{M1}(t)+\lambda J_{M2}^{N}(t))dt,\end{cases}\tag{18}$$

hence the optimal control problems for the manufacturer and the supplier at the pre-recall stage of the product are given by:

$$\begin{cases}\max\limits_{v(t),A(t)}\int_0^{\infty}e^{-(r+\lambda)t}((p-w)D(t)-\frac{1}{2}\eta v^2(t)-\frac{1}{2}hA^2(t)+\lambda V_{M2}^{N}((1-\phi)G,Q))dt\\ s.t.\dot{G}(t)=k_1A(t)+bQ(t)-\varepsilon_1G(t),\ G(0)=G_0,\\ \dot{Q}(t)=\alpha u(t)+\beta v(t)-\delta Q(t),\ Q(0)=Q_0,\end{cases}\tag{19}$$

$$\begin{cases}\max\limits_{u(t)}\int_0^{\infty}e^{-(r+\lambda)t}(wD(t)-\frac{1}{2}\mu u^2(t)-\frac{1}{2}\theta(1-\lambda)u^2(t)-H+\lambda V_{S2}^{N}((1-\phi)G,Q)dt\\ s.t.\dot{G}(t)=k_1A(t)+bQ(t)-\varepsilon_1G(t),\ G(0)=G_0,\\ \dot{Q}(t)=\alpha u(t)+\beta v(t)-\delta Q(t),\ Q(0)=Q_0.\end{cases}\tag{20}$$

Applying the dynamic planning approach, the HJB equation for the manufacturer and supplier at the stage before the product recall are:

$$\begin{aligned}rV_{M1}^{N}(G,Q)=\max_{A,v}[&(p-w)(a-bp+\sigma G)-\frac{1}{2}\eta v^2-\frac{1}{2}hA^2+\frac{\partial V_{M1}^{N}}{\partial G}(k_1A+bQ-\varepsilon G)\\&+\frac{\partial V_{M1}^{N}}{\partial Q}(\alpha u+\beta v-\delta Q)+\lambda(V_{M2}^{N}((1-\phi)G,Q)-V_{M1}^{N}(G,Q))],\end{aligned}\tag{21}$$

$$\begin{aligned}rV_{S1}^{N}(G,Q)=\max_{u}[&w(a-bp+\sigma G)-\frac{1}{2}\mu u^2-\frac{1}{2}\theta(1-\lambda)u^2+\frac{\partial V_{S1}^{N}}{\partial G}(k_1A+bQ-\varepsilon G)\\&+\frac{\partial V_{S1}^{N}}{\partial Q}(\alpha u+\beta v-\delta Q)+\lambda(V_{S2}^{N}((1-\phi)G,Q)-V_{S1}^{N}(G,Q))].\end{aligned}\tag{22}$$

where the superscript N denotes the non-cooperative scenario, V_{M1}^{N}, V_{S1}^{N} denote the optimal value functions of the manufacturer and the supplier at the pre-recall stage. They also satisfy the boundary conditions $\lim\limits_{t\to\infty}re^{-rt}V_{M1}^{N}=0,$ $\lim\limits_{t\to\infty}re^{-rt}V_{S1}^{N}=0.$ The additional terms $V_{M2}^{N}((1-\phi)G,Q)-V_{M1}^{N}(G,Q),V_{S2}^{N}((1-\phi)G,Q)-V_{S1}^{N}(G,Q)$ denote the change of the manufacturer's and the supplier's expected profits from the pre-recall stage to the post-recall stage. By solving the HJB equations (16) and (17), the feedback solution of Stackelberg game in the non-cooperative scenario is obtained. Proposition 2 shows the equilibrium decision making and optimal value function of the firms at the pre-recall stage in the non-cooperative scenario.

Proposition 2 In the non-cooperative scenario, the optimal advertising and secondary quality investments of the manufacturer and the optimal primary quality investments of the supplier at the pre-recall stage are:

$$A_1^{\mathrm{N}} = -\frac{(p-w)\sigma(\lambda\phi-\lambda-r-\varepsilon_2)k_1}{(r+\varepsilon_2)(r+\lambda+\varepsilon_1)h}, \tag{23}$$

$$v_1^{\mathrm{N}} = \frac{\sigma(p-w)(r^2+((-\phi+2)\lambda+\varepsilon_2+\delta)r+\lambda^2+((1-\phi)\delta+\varepsilon_1)\lambda+\delta\varepsilon_2)b\beta}{(r+\varepsilon_2)(r+\lambda+\varepsilon_1)(r+\delta)(r+\lambda+\delta)\eta}, \tag{24}$$

$$u_1^{\mathrm{N}} = \frac{w\sigma b(\delta\lambda\phi+\lambda\phi r-\delta\lambda-\delta r-\delta\varepsilon_2-\lambda^2-2\lambda r-\lambda\varepsilon_1-r^2-r\varepsilon_2)\alpha}{(r+\varepsilon_2)(r+\lambda+\varepsilon_1)(r+\delta)(r+\lambda+\delta)(\lambda\theta-\mu-\theta)}. \tag{25}$$

The optimal value function at the pre-recall stage of manufacturer and supplier are:

$$V_{\mathrm{M1}}^{\mathrm{N}} = \frac{(p-w)\sigma}{r+\lambda+\varepsilon_1}G + \frac{\sigma^2(\mathrm{p}-w)^2k_1^2}{2h(r+\lambda+\varepsilon_1)^2(r+\lambda+\delta)}Q + F_3, \tag{26}$$

$$V_{\mathrm{S1}}^{\mathrm{N}} = \frac{\sigma w}{r+\lambda+\varepsilon_1}G + \frac{(p-w)k_1^2w\sigma^2}{h(r+\lambda+\varepsilon_1)^2(r+\lambda+\delta)}Q + F_4, \tag{27}$$

where $Q^{\mathrm{N}} = \left(Q^N(\tau)-Q_0\right)e^{-\delta(\tau-t)}+Q_0$, $G^{\mathrm{N}} = \left(G^N(\tau)-G_0\right)e^{-\delta(\tau-t)}+G_0$. $Q^{\mathrm{N}}, G^{\mathrm{N}}$ is the optimal path of product quality and product goodwill at the pre-recall stage, respectively. F_3, F_4 expressions are shown in the appendix due to their complexity.

The equilibrium solution in Proposition 2 is the same as that in Proposition 1, which shows that $\frac{\partial u_1^{\mathrm{N}}}{\partial\sigma}>0,\ \frac{\partial \mathrm{v}_1^{\mathrm{N}}}{\partial\sigma}>0,\ \frac{\partial A_1^{\mathrm{N}}}{\partial\sigma}>0.$ when consumers pay more attention to the quality of goods, manufacturers and suppliers would increase their quality and advertising investments in order to improve product quality and goodwill and better satisfy consumers' demand. However, unlike the equilibrium state at the post-recall stage, at the pre-recall stage, the supply chain equilibrium decision is closely related to the likelihood of a recall and the recall shock, and presents that $\frac{\partial u_1^{\mathrm{N}}}{\partial\phi}<0,\ \frac{\partial \mathrm{v}_1^{\mathrm{N}}}{\partial\phi}<0,\ \frac{\partial A_1^{\mathrm{N}}}{\partial\phi}<0.$ specifically, the three decision variables, namely, manufacturers' advertising investments, secondary quality investments, and primary quality investments of suppliers, all decrease with the recall shock. This is because, as the recall shock increases, firms face greater costs of reputational and financial losses, which prompts them to allocate resources more cautiously, weighing short-term profits against long-term development, and to reduce investment in advertising and quality improvement.

Corollary 1: *There exists a threshold* $\phi^* = \frac{\varepsilon_1-\varepsilon_2}{\varepsilon_1+r}$ *,when* $\phi<\phi^*$ *,there is* $\frac{\partial A_1^{\mathrm{N}}}{\partial\lambda}>0.$ *or else* $\frac{\partial A_1^{\mathrm{N}}}{\partial\lambda}<0.$

Corollary 1 suggests that a manufacturer's advertising strategy is closely related to the likelihood of a crisis occurring as well as to the level of crisis damage. Specifically, when firms anticipate that the potential damage from a crisis is below a specific threshold as described above, manufacturers tend to increase their advertising expenditures as the likelihood of a crisis increases. However, when the impact of a crisis exceeds this threshold, i.e., when the potential damage becomes severe, the manufacturer's advertising strategy changes and chooses to reduce advertising investment as the likelihood of a crisis increases. This is because, in the face of small potential damages, manufacturers may consider it an effective strategy to reduce the impact of the crisis on sales by increasing advertising investment to strengthen brand image and consumer confidence. However, when the crisis impact exceeds this threshold, on the one hand, the excessive level of crisis damage leads to a significant cut in the effectiveness of advertising investment, increasing the financial burden of the firm, and on the other hand, the possibility of negative effects of advertising investment makes manufacturers more cautious in advertising investment.

Scenario of cooperation (C)

Cost-sharing contracts exist between the manufacturer and supplier in this subsection, i.e. $\varphi \in (0,1)$, manufacturers bear φ part of supplier's quality improvement costs .Same to the non-cooperative scenario, uses the inverse-order induction method to solve for the optimal quality and advertising investments of supply chain members. The optimal advertising and quality investments decisions of the manufacturer and supplier at the post-recall stage are solved firstly, and then the optimal advertising and quality investments decisions of the manufacturer and supplier at the pre-recall stage are solved.

Post-recall stage of a product recall in cooperative scenario

The optimal control problems of the manufacturer and supplier are as follows:

$$\begin{cases} \max\limits_{v(t),A(t)} \int_{\tau}^{\infty} e^{-rt}((p-w)D(t)-\frac{1}{2}\eta v^2(t)-\frac{1}{2}hA^2(t)-\varphi\frac{1}{2}\mu u^2(t)--\varphi\frac{1}{2}\theta u^2(t))dt \\ s.t. \dot{G}(t)=k_2A(t)+bQ(t)-\varepsilon_2 G(t),\ G(\tau^+)=(1-\phi)G(\tau^-), \\ \dot{Q}(t)=\alpha u(t)+\beta v(t)-\delta Q(t),\ Q(0)=Q_0, \end{cases} \tag{28}$$

$$\begin{cases} \max\limits_{u(t)} \int_{\tau}^{\infty} e^{-rt}(wD(t)-(1-\varphi)\frac{1}{2}\mu u^2(t)-(1-\varphi)\frac{1}{2}\theta u^2(t))dt \\ s.t. \dot{G}(t)=k_2A(t)+bQ(t)-\varepsilon_2 G(t),\ G(\tau^+)=(1-\phi)G(\tau^-), \\ \dot{Q}(t)=\alpha u(t)+\beta v(t)-\delta Q(t),\ Q(0)=Q_0. \end{cases} \tag{29}$$

Applying dynamic planning method, it can be shown that HJB equation for the manufacturer and supplier at the post-recall stage are:

$$\begin{aligned} rV_{\mathrm{M2}}^{\mathrm{C}}(G,Q) = \max_{A,v}[(p-w)(a-p+\sigma G)-\frac{1}{2}\eta v^2-\frac{1}{2}hA^2-\varphi\frac{1}{2}\mu u^2-\varphi\frac{1}{2}\theta u^2 \\ +\frac{\partial V_{\mathrm{M2}}^{\mathrm{C}}}{\partial G}(k_2A+bQ-\varepsilon_2 G)+\frac{\partial V_{\mathrm{M2}}^{\mathrm{C}}}{\partial Q}(\alpha u+\beta v-\delta Q)], \end{aligned} \tag{30}$$

$$\begin{aligned} rV_{\mathrm{S2}}^{\mathrm{C}}(G,Q) = \max_{u}[w(a-p+\sigma G)-(1-\varphi)\frac{1}{2}\mu u^2-(1-\varphi)\frac{1}{2}\theta u^2 \\ +\frac{\partial V_{\mathrm{S2}}^{\mathrm{C}}}{\partial G}(k_2A+bQ-\varepsilon_2 G)+\frac{\partial V_{\mathrm{S2}}^{\mathrm{C}}}{\partial Q}(\alpha u+\beta v-\delta Q)], \end{aligned} \tag{31}$$

where the superscript C denotes the cooperative scenario, $V_{\mathrm{M2}}^{\mathrm{C}}, V_{\mathrm{S2}}^{\mathrm{C}}$ respectively denote the optimal value functions of the manufacturer and supplier at the post-recall stage, and the boundary conditions $\lim\limits_{t\to\infty} re^{-rt}V_{\mathrm{M2}}^{\mathrm{C}}=0$ and $\lim\limits_{t\to\infty} re^{-rt}V_{\mathrm{S2}}^{\mathrm{C}}=0$ are satisfied. By solving the HJB equations (20) (21) to obtain the feedback solution of the Stackelberg game in the cooperative scenario, Proposition 3 shows the equilibrium decision and optimal value function of the firms at the post-recall stage in the cooperative scenario.

Proposition 3 In the cooperative scenario, the optimal advertising and secondary quality investments of the manufacturer and the optimal primary quality investments of the supplier at the post-recall stage are:

$$A_2^{C} = \frac{(p-w)\sigma k_2}{(r+\varepsilon_2)h}, \tag{32}$$

$$v_2^{C} = \frac{(p-w)\sigma b\beta}{(r+\varepsilon_2)(r+\delta)\eta}, \tag{33}$$

$$u_2^{C} = -\frac{bw\sigma\alpha}{(r+\varepsilon_2)(r+\delta)(\varphi\mu+\varphi\theta-\mu-\theta)}. \tag{34}$$

The best-value function for the manufacturer and supplier at the post-recall stage in the cooperative scenario are:

$$V_{M2}^{C} = \frac{(p-w)\sigma}{r+\varepsilon_2}G + \frac{(p-w)\sigma b}{(r+\varepsilon_2)(r+\delta)}Q + F_5, \tag{35}$$

$$V_{S2}^{C} = \frac{w\sigma}{(r+\varepsilon_2)}G + \frac{bw\sigma}{(r+\varepsilon_2)(r+\delta)}Q + F_6, \tag{36}$$

where $Q^{C} = (Q_0 - Q_\infty^{C})e^{-\delta t} + Q_\infty^{C}, G^{C} = (G_0 - G_\infty^{C})\mathrm{e}^{-\varepsilon_2 t} + G_\infty^{C}$ are the optimal paths for product quality and goodwill at that stage. The steady state of product quality and goodwill are $Q_\infty^{C} = \frac{b\sigma(-\beta^2\mu p\varphi + \beta^2\mu\varphi w - \beta^2 p\theta\varphi + \beta^2\theta\varphi w + \alpha^2 w\eta + \beta^2\mu p - \beta^2\mu w + \beta^2 p\theta - \beta^2\theta w)}{\delta(r+\varepsilon_2)(r+\delta)(\varphi-1)(\mu+\theta)\eta}$, $G_\infty^{C} = \frac{bQ_\infty^{C}hr + bQ_\infty^{C}h\varepsilon_2 + \sigma k_2^2 p - \sigma k_2^2 w}{\varepsilon_2(r+\varepsilon_2)h}$, respectively. The F_5 and F_6 expression shows in the appendix due to their complexity.

Following Proposition 1, manufacturers' optimal strategies in terms of advertising investments and secondary quality investments, as well as suppliers' optimal strategies in terms of primary quality investments, are independent of the likelihood of product recalls and the extent of crisis damage, and are positively and positively related to the market demand sensitivity to goodwill. This implies that as market sensitivity to goodwill rises, these optimal strategies strengthen accordingly. It is worth noting that the manufacturer's optimal secondary quality investment strategy and optimal advertising investment strategy under the cost-sharing contract are consistent with the no contract scenario ($v_2^C = v_2^N, A_2^C = A_2^N$), however, this contractual mechanism positively affects the supplier's primary quality investment decision ($u_2^C > u_2^N$). In the cooperative situation, the manufacturer effectively reduces the burden on the supplier through the cost-sharing contract, which makes the supplier willing to invest more primary quality effort in the cooperative situation than in the non-cooperative situation. This cooperative mechanism leads to the production of higher quality products than in the non-cooperative situation.

Then why did the manufacturer not change its advertising and quality input strategies under the cost-sharing contract? The reason is that the cooperation between manufacturers and suppliers is complementary. While the manufacturer's advertising and quality inputs are designed to enhance the overall competitiveness and market presence of the product, the supplier's primary quality investment is key to ensuring the basic quality of the product. Under the cost-sharing contract, although suppliers have more resources to invest in primary quality, it does not mean that manufacturers can neglect their own advertising and quality inputs. This is because only when the manufacturer's advertising and quality inputs are coordinated with the supplier's quality investment can the overall quality and goodwill of the product be maximized, so as to achieve the overall optimization of the supply chain and maximize profitability. Therefore, manufacturers keep their advertising and quality investment strategies unchanged under the cost-sharing contract in order to ensure that the cooperation with suppliers can generate maximum synergies and jointly enhance the competitiveness and market position of their products.

Pre-recall stage of a product recall in a cooperative scenario

The expected profits of suppliers and manufacturers denote by $\Pi_{S1}^{C}(t)$, $\Pi_{M1}^{C}(t)$:

$$\begin{cases}\Pi_{S1}^{C}(t)=E_t[\int_0^{\tau}e^{-rs}\pi_{S1}(s)ds+e^{-r\tau}J_{S2}^{C}(\tau)],\\ \Pi_{M1}^{C}(t)=E_t[\int_0^{\tau}e^{-rs}\pi_{M1}(s)ds+e^{-r\tau}J_{M2}^{C}(\tau)],\end{cases}\tag{37}$$

where $J_{M2}^{C}(\tau)=\int_{\tau}^{\infty}e^{-rt}\pi_{M2}^{C}(t)dt, J_{S2}^{C}(\tau)=\int_{\tau}^{\infty}e^{-rt}\pi_{S2}^{C}(t)dt$, respectively denote the net present value of the manufacturer's and supplier's profits at moment τ. Equation (22) can be transformed according to the integration of divisions and algebraic operations:

$$\begin{cases}\Pi_{S1}^{C}(t;\lambda)=\int_0^{\infty}e^{-(r+\lambda)t}(\pi_{S1}^{C}(t)+\lambda J_{S2}^{C}(t)dt,\\ \Pi_{M1}^{C}(t;\lambda)=\int_0^{\infty}e^{-(r+\lambda)t}(\pi_{M1}^{C}(t)+\lambda J_{M2}^{C}(t))dt.\end{cases}\tag{38}$$

Applying dynamic planning method, it can be shown that the optimal control problem of the manufacturer and supplier at the pre-recall stage are:

$$\begin{cases}\max\limits_{v(t),A(t)}\int_0^{\infty}e^{-(r+\lambda)t}(p-w)D(t)-\frac{1}{2}\eta v^2(t)-\frac{1}{2}hA^2(t)-\varphi\frac{1}{2}(\mu-\theta(1-\lambda))u^2(t)+\lambda(V_{M2}^{N}((1-\phi)G,Q))dt\\ s.t.\dot{G}(t)=k_1A(t)+bQ(t)-\varepsilon_1G(t),\ G(0)=G_0,\\ \dot{Q}(t)=\alpha u(t)+\beta v(t)-\delta Q(t),\ Q(0)=Q_0,\end{cases}\tag{39}$$

$$\begin{cases}\max\limits_{u(t)}\int_0^{\infty}e^{-(r+\lambda)t}(wD(t)-(1-\varphi)\frac{1}{2}\mu u^2(t)-(1-\varphi)\frac{1}{2}\theta(1-\lambda)u^2(t)+\lambda V_{S2}^{N}((1-\phi)G,Q)dt\\ s.t.\dot{G}(t)=k_1A(t)+bQ(t)-\varepsilon_1G(t),\ G(0)=G_0,\\ \dot{Q}(t)=\alpha u(t)+\beta v(t)-\delta Q(t),\ Q(0)=Q_0.\end{cases}\tag{40}$$

Hence the HJB equation of the manufacturer and supplier at the stage before product recall are:

$$rV_{M1}^{C}(G,Q)=\max_{A,v}[(p-w)D-\frac{1}{2}\eta v^2-\frac{1}{2}hA^2-\varphi\frac{1}{2}\mu u^2-\varphi\frac{1}{2}\theta(1-\lambda)u^2+\frac{\partial V_{M1}^{C}}{\partial G}(k_1A+ bQ-\varepsilon_1G)+\frac{\partial V_{M1}^{C}}{\partial Q}(\alpha u+\beta v-\delta Q)+\lambda(V_{M2}^{C}((1-\phi)G,Q)-V_{M1}^{C}(G,Q))],\tag{41}$$

$$rV_{S1}^{C}(G,Q)=\max_{u}[wD-(1-\varphi)\frac{1}{2}\mu u^2-(1-\varphi)\frac{1}{2}\theta(1-\lambda)u^2+\frac{\partial V_{S1}^{C}}{\partial G}(k_1A+bQ -\varepsilon_1G)+\frac{\partial V_{S1}^{C}}{\partial Q}(\alpha u+\beta v-\delta Q)+\lambda(V_{S2}^{C}((1-\phi)G,Q)-V_{S1}^{C}(G,Q))],\tag{42}$$

where the superscript C denotes the cooperative scenario, V_{M1}^{C}, V_{S1}^{C} denote the optimal value functions of the manufacturer and the supplier at the post-recall stage, and satisfy the boundary conditions $\lim_{t\to\infty} re^{-rt}V_{M1}^{C}=0, \lim_{t\to\infty} re^{-rt}V_{S1}^{C}=0$. The additional terms denote the changes of the manufacturer's and supplier's expected profits from the pre-recall stage to the post-recall stage, i.e. $V_{M2}^{C}((1-\phi)G,Q)-V_{M1}^{C}(G,Q)$, $V_{S2}^{C}((1-\phi)G,Q)-V_{S1}^{C}(G,Q)$. By solving the above HJB equations (26) (27), the feedback solution of Stackelberg game in the cooperative scenario is obtained, and Proposition 4 shows the equilibrium decision-making of the firms and the optimal value function at the pre-recall stage in the cooperative scenario.

Proposition 4 In the cooperative scenario, the manufacturer's optimal advertising and secondary quality investments and the supplier's optimal primary quality investments at the pre-recall stage are:

$$A_1^{C}=-\frac{(p-w)\sigma(\lambda\phi-\lambda-r-\varepsilon_2)k_1}{(r+\varepsilon_2)(r+\lambda+\varepsilon_1)h}, \tag{43}$$

$$v_1^{C}=-\frac{(p-w)\sigma b(\delta\lambda\phi+\lambda\phi r-\delta\lambda-\delta r-\delta\varepsilon_2-\lambda^2-2\lambda r-\lambda\varepsilon_1-r^2-r\varepsilon_2)\beta}{(r+\varepsilon_2)(r+\lambda+\varepsilon_1)(r+\delta)(r+\lambda+\delta)\eta}, \tag{44}$$

$$u_1^{C}=\frac{bw\sigma(r^2+((-\phi+2)\lambda+\delta+\varepsilon_2)r+\lambda^2+((1-\phi)\delta+\varepsilon_1)\lambda+\delta\varepsilon_2)\alpha}{(r+\varepsilon_2)(r+\lambda+\varepsilon_1)(r+\delta)(r+\lambda+\delta)(\lambda\theta\varphi-\lambda\theta-\mu\varphi-\theta\varphi+\mu+\theta)}. \tag{45}$$

The optimal value function of the manufacturer and supplier products at the post-recall stage in the cooperative scenario are:

$$V_{M1}^{C}=\frac{(p-w)\sigma(\lambda+r+\varepsilon_2-\lambda\phi)}{(r+\varepsilon_2)(r+\lambda+\varepsilon_1)}G+\frac{(p-w)\sigma b(\delta\lambda+\delta r+\delta\varepsilon_2+\lambda^2+2\lambda r+\lambda\varepsilon_1+r^2+r\varepsilon_2-\delta\lambda\phi-\lambda\phi r)}{(r+\varepsilon_2)(r+\lambda+\varepsilon_1)(r+\delta)(r+\lambda+\delta)}Q+F_7, \tag{46}$$

$$V_{S1}^{C}=\frac{w\sigma(\lambda+r+\varepsilon_2-\lambda\phi)}{(r+\varepsilon_2)(r+\lambda+\varepsilon_1)}G+\frac{bw\sigma(r^2+((2-\phi)\lambda+\delta+\varepsilon_2)r+\lambda^2+((1-\phi)\delta+\varepsilon_1)\lambda+\delta\varepsilon_2)}{(r+\varepsilon_2)(r+\lambda+\varepsilon_1)(r+\delta)(r+\lambda+\delta)}Q+F_8, \tag{47}$$

where $Q^{C}=\left(Q^{C}(\tau)-Q_0\right)e^{-\delta(\tau-t)}+Q_0, G^{C}=\left(G^{C}(\tau)-G_0\right)e^{-\delta(\tau-t)}+G_0$. Q^{C}, G^{C} are optimal paths of product quality and goodwill. F_7, F_8 expressions shows in the appendix due to their complexity.

Proposition 2 is further echoed by revealing that the manufacturer's optimal strategy in terms of advertising investments and secondary quality investments, as well as the supplier's optimal strategy in terms of primary quality investments, are positively related to the market demand sensitivity to goodwill. This implies that as the market sensitivity to goodwill increases, supply chain members will increase the corresponding investment to enhance the alloying power of product goodwill. Meanwhile, these equalization strategies are negatively related to the degree of crisis damage, suggesting that manufacturers and suppliers would be more prudent in formulating advertising and quality investment strategies in the face of a potential crisis. Moreover, consistent with Proposition 3, the optimal level of secondary quality investments as well as the optimal advertising investment strategy of manufacturers in the pre-crisis phase are equal to those in the no-contract scenario under the cost-sharing contract ($v_1^{C}=v_1^{N}$, $A_1^{C}=A_1^{N}$), and the level of primary quality investments of suppliers under the cost-sharing contract is greater than that in the no-contract scenario ($u_1^{C}>u_1^{N}$). This implies that the cost-sharing contract does not change the manufacturer's pre-crisis investment strategy and that the cost-sharing contract effectively incentivizes the supplier to increase primary quality investments, which in turn improves the overall quality of the product. We further analyze that suppliers' primary quality investments are positively related to cost-sharing ratios($\frac{\partial u_1^{C}}{\partial\varphi}>0, \frac{\partial u_2^{C}}{\partial\varphi}>0$). As the cost-sharing ratio set by the manufacturer increases, the cost burden of the supplier gradually decreases, so that the supplier has more money for primary quality investments. This cost-sharing mechanism not only reduces the economic pressure on suppliers, but also motivates them to invest more in primary quality efforts. Therefore, through a reasonable cost-sharing contract design, manufacturers can effectively incentivize suppliers to increase primary quality inputs and improve the overall quality and competitiveness of their products.

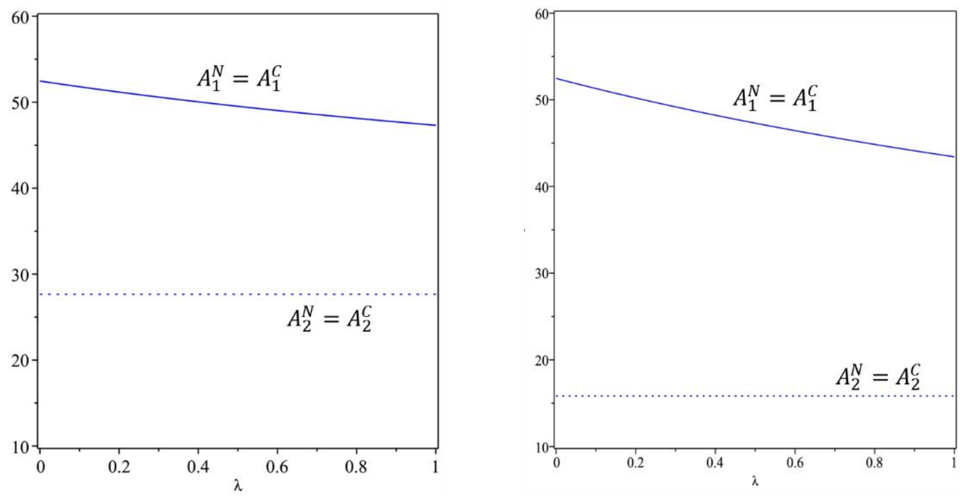
Equilibrium Analysis

In order to achieve long-term stable development, forward-thinking managers pay close attention to two key factors and flexibly adjust their strategic layout according to their changes: one is the likelihood of product recalls and the other is the degree of damage caused by product recalls. In this subsection, we use numerical simulation to analyze how the different characteristics of recall events affect the equilibrium decision and equilibrium profit of a firm by assigning different values to these two factors. In order not to lose generality referring to the setting of Mukherjee et al (2023), the benchmark parameters are set as follows: $\mu=\theta=5,\eta=h=15,\alpha=9,\beta=15$, $\sigma=12,\delta=2,r=0.05,p=30,a=100,w=10$. We assume high recall likelihood ($\lambda=0.5$) , low recall likelihood ($\lambda=0.1$) , high recall shock ($\phi=0.5$) , low recall shock ($\phi=0.2$) , and the values of the corresponding goodwill impact parameters are shown in Table 3.

Table 3: Values of goodwill influencing factors corresponding to recall shock degree

degree of recall shock ϕ	The values of k_i	The values of ε_i
High ($\phi=0.5$)	$k_1=10,k_2=5$	$\varepsilon_1=3,\varepsilon_2=5$
Low ($\phi=0.2$)	$k_1=10,k_2=7$	$\varepsilon_1=3,\varepsilon_2=4$

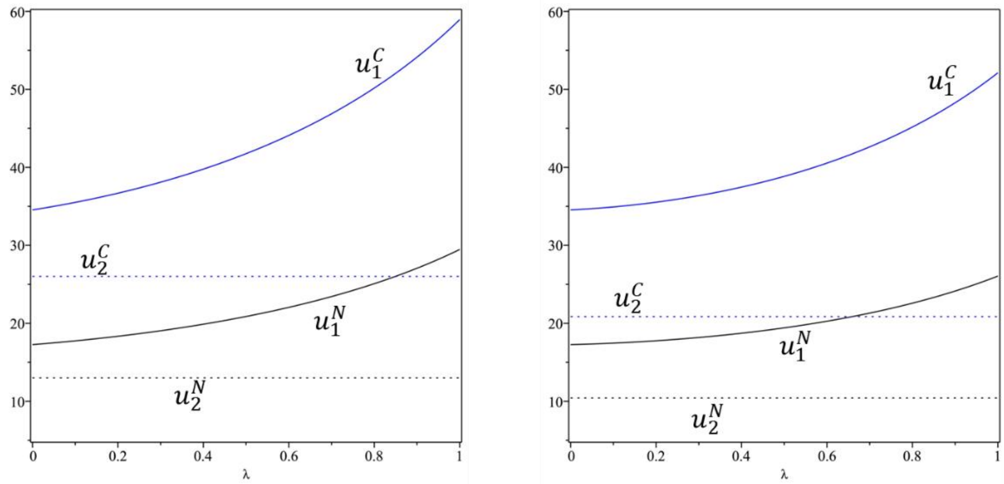
The effect of recall likelihood and the extent of recall damage on equilibrium decisions



（a） $\phi=0.2$ (b) $\phi=0.5$

Figure 1: λ & ϕ impact on manufacturers' advertising investment decisions

Figure 1 reveals the following phenomenon: regardless of whether the level of recall damage is high or low, a manufacturer's advertising investment in the pre-recall phase decreases as the likelihood of a recall occurring increases. Moreover, manufacturers' equilibrium advertising investment function declines significantly faster when recall damage is high. This trend suggests that manufacturers prudently adjust their advertising investment strategies in the face of potential product recall costs, especially when recall damages are expected to be high, and that manufacturers reduce their advertising investment more rapidly to cope with the possible negative impacts.

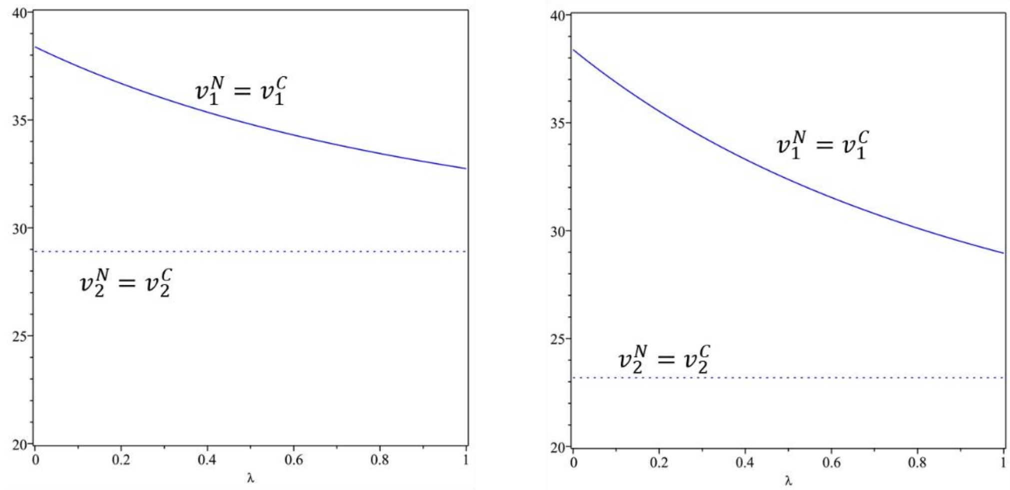


（a） $\phi = 0.2$ (b) $\phi = 0.5$

Figure 2: λ & ϕ impact on Suppliers' primary quality investment decisions

Figure 2 clearly demonstrates the trend that suppliers' balanced primary quality investment decisions prior to a recall are positively correlated with the likelihood of a crisis occurring under different scenarios of high or low recall damage. In other words, when the cost of a product recall increases, suppliers increase their primary quality inputs accordingly, aiming to prevent potential product recalls by improving product quality. This strategy reflects the high importance that suppliers place on product quality and market reputation.

However, it is worth noting that the rate of increase in the supplier's equilibrium primary quality input function slows down when the recall damage level is high. This may be due to the fact that in high damage scenarios, suppliers face greater economic pressure and uncertainty, leading them to be more cautious in increasing their primary quality inputs. In addition, high damage scenarios may also prompt suppliers to make finer trade-offs in resource allocation to ensure that quality improvements are made while also taking into account cost control and overall economic efficiency. Such strategic adjustments reflect the complex decision-making process of suppliers in responding to the cost of a high-damage recall.

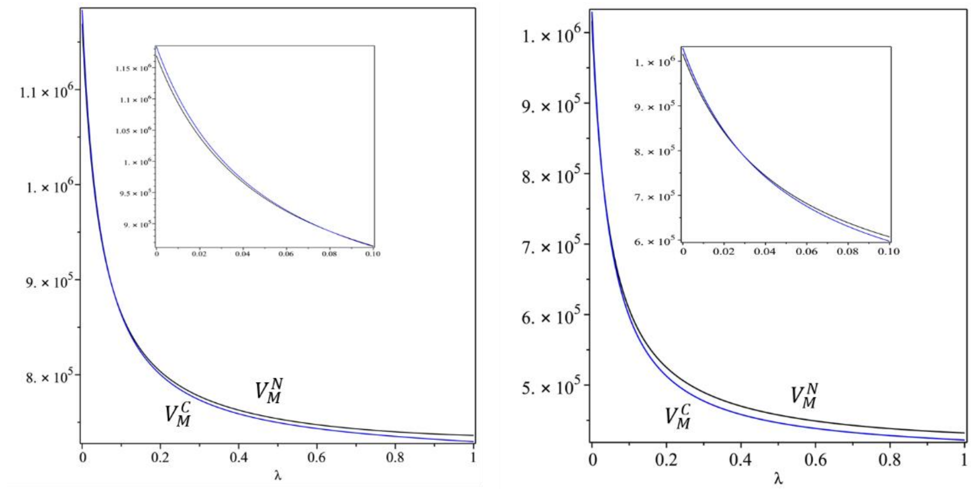


（a） $\phi = 0.2$ (b) $\phi = 0.5$

Figure 3: λ & ϕ impact on manufacturers' sub-quality investment decisions

Figure 3 further reveals the relationship between a manufacturer's primary quality investment decisions in the pre-recall phase and the likelihood of a recall occurring. Regardless of whether the level of recall damage is high or low, the manufacturer's primary quality input decision decreases as the likelihood of recall increases. This phenomenon suggests that manufacturers may adjust their primary quality input strategies in the face of potential product recall costs. Notably, the rate of decrease in the manufacturer's primary quality equilibrium function in the pre-recall phase accelerates significantly when the level of recall damage is high. This may be due to the fact that, when faced with a high level of recall damage, manufacturers will be more inclined to reduce their primary quality inputs in order to mitigate the potential economic burden. However, this strategy may require a trade-off between product quality and cost control to ensure that recall cost is reduced without disproportionately affecting product quality.

The effect of recall likelihood and the extent of recall damage on equilibrium profits

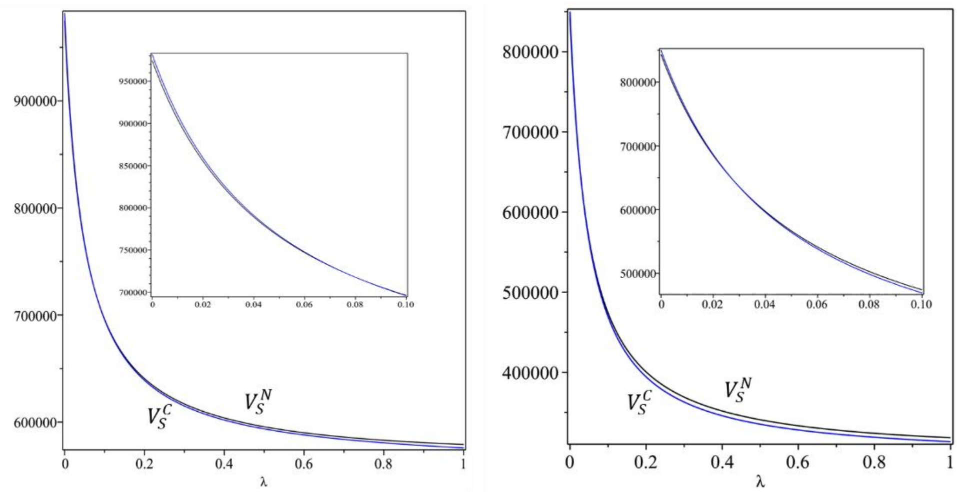


（a） $\phi = 0.2$ (b) $\phi = 0.5$

Figure 4: λ & ϕ impact on Manufacturers' Equilibrium Profits

Figure 4(a) illustrates the manufacturer's profit function for the non-cooperative and cooperative scenarios, which intersect at one point. When the predicted likelihood of recall occurrence is lower than this intersection point, the manufacturer's profit is significantly higher in the cooperative scenario than in the non-cooperative scenario. In this scenario, manufacturers are more likely to enter into cost-sharing contracts with suppliers to more effectively coordinate the supply chain to address potential recall costs. However, once the likelihood of a recall exceeds this intersection, manufacturer profits in the non-cooperative scenario instead exceed those in the cooperative scenario, at which point the manufacturer lacks incentives to cooperate with the supplier.

Turning to Figure 4(b), we observe that manufacturers' profit functions in the non-cooperative and cooperative scenarios similarly intersect at one point. However, it is worth noting that the intersection point in Figure (b) corresponds to a significantly lower likelihood of recall occurrence than the intersection point in Figure (a). This phenomenon reveals that the manufacturer's sensitivity to the likelihood of recall occurrence increases when the recall damage level increases. In this case, manufacturers may be more inclined to adopt a conservative strategy because they perceive the market environment to be full of uncertainty and instability. As a result, manufacturers may be more inclined to self-protect in the face of a higher cost of recall damage than to seek cooperation with their suppliers. This trend highlights manufacturers' strategic choices and cost preferences in the face of different levels of recall damage.



（a） $\phi = 0.2$ (b) $\phi = 0.5$

Figure 5: λ & ϕ impact on Suppliers' Equilibrium Profits

Figure 5 clearly shows the trend of the supplier's profit function in the cooperative and non-cooperative scenarios. As the likelihood of recalls increases, the profit functions in both scenarios trend downward and intersect at a certain point. Specifically, when the manufacturer's predicted likelihood of a product recall occurring is below this intersection point, the supplier's profit is higher in the cooperative than in the non-cooperative scenario. This means that in this case, suppliers have an incentive to cooperate with the manufacturer. However, when the likelihood of recalls exceeds this intersection, suppliers are more profitable in the non-cooperative context.

It is often assumed that in the cooperative scenario, the manufacturer helps the supplier to bear some of the cost pressure, thus motivating the supplier to prefer to cooperate with the manufacturer. However, our analysis shows that the reality is not so simple. Combining the observations in Figures 2 and 3, we can see that manufacturers progressively reduce secondary quality inputs as the likelihood of recalls increases, while suppliers progressively increase primary quality inputs. In our model, product quality is jointly determined by the joint quality efforts of both parties. When the positive utility of the supplier's increase in primary quality input is not sufficient to offset the negative impact of the manufacturer's decrease in secondary quality input, product quality may decrease rather than increase. Thus, although the manufacturer's help in a cooperative situation may appear to be beneficial to the supplier, in some cases such cooperation may not always be beneficial to the supplier's profit maximization. When deciding whether to partner with a manufacturer, suppliers need to consider the likelihood of a recall, the quality input strategies of both parties, and the impact of product quality on profits. This combination of considerations makes the supplier's decision when faced with the cost of a recall more complex and nuanced.

Conclusion

Based on the research in this paper, the following managerial insights are obtained: (1) whether in cooperative or non-cooperative situations, the pre-recall stage corporate advertising and quality investment decisions are affected by the recall likelihood, and manufacturers predicting a higher likelihood of recall will reduce the corresponding advertising and quality inputs to save costs and to prepare for the necessary expenditures in the post recall stage of the product; and the post-recall stage corporate advertising and quality investment decisions are independent of the recall possibility. (2) When consumers pay more attention to product quality, firms would increase their advertising and quality inputs, even though this would increase the corresponding marketing and operating costs. (3) Manufacturers can effectively incentivize suppliers to make more primary quality inputs and improve product quality through cost-sharing contracts. (4) The effectiveness of the cost-sharing contract in coordinating the supply chain is affected by the likelihood of recalls and the extent of recall damage, and cooperation can only be achieved when both are relatively small. In addition, this paper considers the quality and advertising strategy adjustments made by supply chain members in response to product recalls, exogenizing price, and pricing strategies. This paper considers the role of cost-sharing contracts in supply chain coordination, and may also consider the coordination role of other types of contracts.

References

1. Liu Y, Shankar V, Yun W (2017) Crisis Management Strategies and the Long-Term Effects of Product Recalls on Firm Value. Journal of Marketing. 81(5): 30-48.
2. Wei J C, Zhao M, Wang G, Zhao D T (2016), The Effects of firm Actions on Customers Responses to Product Recall Crises: Analyzing an Automobile Recall in China. Journal of Cost Research. 19(4): 1-19.
3. Shah R, Ball G P, Netessine S (2017), Plant Operations and Product Recalls in the Automotive Industry: An Empirical investigation. Management Science, 63(8): 2439-2459.
4. Eilert M, Jayachanran S, Kalaignanam K, Swart T A (2017), Does It Pay to Reall Your Product Early? An Empirical Investigation in the Automobile Industry. Journal of Marketing. 81(3). 111-129.
5. Borah A, Tellis G J (2016). Halo (spillover) effects in social media: Do product recalls of one brand hurt or help rival brands? Journal of Marketing Research, 53(2): 143-160.
6. Jue Lu, Jorge Navas (2021). Advertising and quality improving strategies in a supply chain when facing potential crises. European Journal of Operational Research, 288: 839-851.
7. Tulika Chakraborty, Arka Mukherjee, Satyaveer S, Chauhan (2023). Should a powerful manufacturer collaborate with a costy supplier? Pre-recall vs. post-recall strategies in product harm crisis management. Computers & Industrial Engineering, 177: 109-137.
8. Cleeren K, Dekimpe M G, Van Heerde (2017). Marketing research on product-harm crises: A view of managerial implications and an agenda for future research. Journal of the Academy of Marketing Science. 45 (5): 593-615.
9. Lister R (2010). Toyota shares plummet following US pressure. URL: http: //news.bbc.co.uk /2/hi/business /8497448.stm.
10. Dong Y, Xu K, Song S, Zhou C (2021). Quality reference in consumer product recalls. Decision Sciences. 52 (4): 893-919.
11. Pressman A (2017). Samsung Smartphone Sales Plummeted Worldwide Due to Galaxy Note 7. https: //fortune.com/2017/01/31/Samsung-galaxy-note-7-recall-apple/.
12. Kim T (2018). Tesla shares drop after its big model recall and more questions about autopilot. https://www.cnbc.com/2018/04/02/tesla-shares-are-tanking-for-several-reasons-including-a-big -recall-and-more-questions-about-autopilot. html.
13. Yoo S H, Cheong T (2018). Quality improvement incentive strategies in a supply chain. Transportation Research Part E: Logistics and Transportation Review.114:331-342.
14. Buratto A, Cesaretto R, De Giovanni P (2019). Consignment contracts with cooperative programs and price discount mechanisms in a dynamic supply chain. International Journal of Production Economics,.218: 72-82.
15. El Ouardighi F (2014). Supply quality management with optimal wholesale price and revenue sharing contracts: A two-stage game approach. International Journal of Production Economics. 156: 260-268.
16. Yong Long, Deng Xu, Xiao xiao Li (2019). Channel coordination of battery supplier and battery swap station of micro-grid with uncertain rental demand. Soft Computing 23:9733-9745.
17. Mohammad AlipourVaezi, Amir Aghsami, Masoud Rabbani (2022). Introducing a novel revenue-sharing contract in media supply chain management using data mining and multi-criteria decision-making methods. Soft Computing 26:2883-2900.
18. Nana Ma, Rong Gao, Xiao bin Wang, Ping Li (2020). Green supply chain analysis under cost sharing contract with uncertain information based on confidence level. Soft Computing 24:2617-2635.

19. Jia guo Liu, Huan Zhou, Jun jin Wang (2018). The coordination mechanisms of emergency inventory model under supply disruptions. Soft Computing 22:5479-5489.
20. Selwyn Piramuthu, Poorya Farahani, Martin Grunow (2013). RFID-generated traceability for contaminated product recall in perishable food supply networks. European Journal of Operational Research.225:253-262.
21. Freddy Moises Brofman Epelbaum, Marian Garcia Martinez (2014). The technological evolution of food traceability systems and their impact on firm sustainable performance: A RBV approach. International Journal of Production Economics.150:215-224.
22. Hongyan Dai, Ling Ge, Weihua Zhou (2015). A design method for supply chain traceability systems with aligned interests. International Journal of Production Economics.170:14-24.
23. Wen li Wang, Jian wen Luo (2015). Optimal financial and ordering decisions of a firm with insurance contract. Technological and Economic Development of Economy.21:257-279.
24. Ling xiu Dong, Brian Tomlin (2012). Managing Disruption Cost: The Interplay Between Operations and Insurance. Management Science.58(10): 1898-1915.
25. Juan Camilo Serpa, Harish Krishnan (2016). The Strategic Role of Business Insurance. Management Science.63:384-404.
26. Lopez M (2017). Samsung Explains Note 7 Battery Explosions and Turns Crisis into Opportunity. https: //www.forbes.com/sites/maribellopez/2017/01/22/
27. Fenech S (2017). Samsung kicks off ad campaign to regain customer trust after Note7 recall.http://www.techguide.com.au/news/samsung-kicks-off-ad-campaign-regain-customer-trust -note7-recall/ .
28. Arka Mukherjee, Margarida Carvalho, Georges Zaccour (2023), Managing quality and pricing during a product recall: An analysis of pre-crisis, crisis and post-crisis regimes. European Journal of Operational Research. 307: 406-420.
29. Rubel O (2018). Profiting from product-harm crises in competitive markets. European Journal of Operational Research, 265 (1): 219-227.
30. Helena Perrone, Rosa Ferrer (2022) Consumers Costly Responses to Product-Harm Crises. Management Science 69(5): 2639-2671.
31. Chauhan S S, Chakraborty T, Ouhimmou M (2019). Cost-sharing mechanism for product quality improvement in a supply chain under competition. International Journal of Production Economics. 208: 566-587.
32. De Giovanni P (2019). An optimal control model with defective products and goodwill damages. Annals of Operations Research. 1-12.
33. Pietro De Giovanni, Georges Zaccour (2023). A survey of dynamic models of product quality. European Journal of Operational Research. 307: 991-1007.